Chapter 3: Adjoints in Euclidean Spaces and Orthogonal Transformations

Exercise 22. Let $n \geq 3$. Consider the matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ defined by

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Let us denote by $f : \mathbb{R}^n \to \mathbb{R}^n$ the endomorphism of $\mathbb{R}^n$ associated to A (in the canonical basis of $\mathbb{R}^n$).

1. Prove that A is diagonalizable.
2. Determine the rank of f and compute $\dim\ker(f)$.
3. Prove that $\ker(f) \oplus \mathrm{im}(f) = \mathbb{R}^n$.
4. Set $F = \mathrm{im}(f)$. Prove that F is stable under f. Prove that $f\big|_F \in \mathrm{End}(F)$ is diagonalizable.
5. Determine a basis $\mathscr{B}$ of F, and compute $B := \mathrm{M}\left(f\big|_F\right)_{\mathscr{B}}$.
6. Diagonalize B.
7. Diagonalize A.

Solution of exercise 1.

1. ${}^tA = A$, by theorem 3.4, A is diagonalizable.
2. The rank of f is 2 because column 1 and column 2 are linearly independent, and they span the rest columns. $\dim\ker(f) = n - \mathrm{rank}(f) = n - 2$.
3. It is not always true for "inner" direct sums, *e.g.* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. By theorem 3.4, there exists a diagonal matrix D, an orthogonal matrix P such that $A = PD{}^tP$, after reordering the eigenvectors that form P, we assume the first two columns of P corresponding to nonzero eigenvalues and they span $\mathrm{Im}(f)$, the rest columns correspond to the $\mathrm{Ker}(f)$.
4. Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \in \mathbb{R}^n$, $Ax = \begin{pmatrix} \sum_i x_i \\ x_1 + x_n \\ \vdots \\ x_1 + x_n \\ \sum_i x_i \end{pmatrix}$, therefore $\mathrm{Im}(f) = \mathrm{Span}(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix})$, denote these two vectors by w_1, w_2 respectively. We have $Aw_1 = 2(w_1 + w_2) \subset \mathrm{Im}(f)$ and $Aw_2 = (n-2)w_1 \subset \mathrm{Im}(f)$. $f\big|_F \in \mathrm{End}(F) = \begin{pmatrix} 2 & n-2 \\ 2 & 0 \end{pmatrix}$ which is clearly diagonalizable. Or prove the fact 'Diagonalizable transformation restricted to an invariant subspace is diagonalizable'.

5. Done.
6. Assume $v_\lambda = w_1 + \lambda w_2$ is an eigenvector of $f\big|_F$, then Aw is colinear with w forces: $(n-2)\lambda^2 + 2\lambda - 2 = 0$, solve it, we get $\lambda = \frac{-1\pm\sqrt{1+2(n-2)}}{n-2}$. v_λ is an eigenvector associated with eigenvalues $2/\lambda = 1 \pm \sqrt{1+2(n-2)}$. (**Warning:** not with eigenvalue λ!)

   Remark. **Sanity check**: Surely, the eigenvalues can be found by finding the eigenvalues of $\begin{pmatrix} 2 & n-2 \\ 2 & 0 \end{pmatrix}$, which are the same. This can verify that our calculation is correct.

7. Diagonalize A by adding eigenvector associated with eigenvalue 0, by letting $Ax = \begin{pmatrix} \sum_i x_i \\ x_1 + x_n \\ \vdots \\ x_1 + x_n \\ \sum_i x_i \end{pmatrix} = 0$. They are $e_2 - e_3, \ldots, e_{n-2} - e_{n-1}, e_1 - e_n$.

   Remark. **Sanity check**: We know that eigenvectors of self-adjoint operator corresponding to different eigenvalues are orthogonal to each other. Let us see in this case: $v_\lambda = \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda \\ 1 \end{pmatrix}$, we have $\langle v_\lambda, e_2 - e_3 \rangle = 0$, $\ldots$, $\langle v_\lambda, e_{n-2} - e_{n-3} \rangle = 0$, which is to say eigenvectors associated with $1 \pm \sqrt{1+2(n-2)}$ ($\neq 0$ when $n \geq 3$) are respectively orthogonal with eigenvectors associated with 0. The eigenvectors associated with 0 we listed above are not 2-by-2 orthogonal. The two v_λ are 2-by-2 orthogonal: $\langle v_{\lambda_1}, v_{\lambda_2} \rangle = 2 + (n-2)\lambda_1\lambda_2 = 2 + (n-2)\dfrac{-2}{n-2} = 0$.

□